

Week of September 22nd

Additional Exercises

Real and Complex Numbers Reciprocal spaces, X-ray diffraction

Exercise 1: irrationality of $\sqrt{2}$ and \sqrt{n}

1a. We want to show that $\sqrt{2}$ is irrational, or in other words, there is no $(n, m) \in \mathbb{N}^2$, with $\gcd(n, m) = 1$, such that $2 = \left(\frac{m}{n}\right)^2$.

- (i) Show that 2 divides m.
- (ii) Show then that 2 divides n.
- (iii) Express a contradiction with $\gcd(n, m) = 1$.

1b. We want to show that $\forall n \in \mathbb{N}, (\sqrt{n} \in \mathbb{R} \setminus \mathbb{Q} \text{ or } \sqrt{n} \in \mathbb{N})$. We consider $n \in \mathbb{N}$ and suppose that $\sqrt{n} \in \mathbb{Q}$. Hence, $\exists (a, b) \in \mathbb{N}^2, \sqrt{n} = \frac{a}{b}$ and $\gcd(a, b) = 1$. Consider a prime number p in the factorization of b :

- (i) Show that $p|a^2$, and hence necessarily we must have $p|a$.
- (ii) Show that $b = 1$ and conclude.

Exercise 2 : Inequalities of Cauchy-Schwartz and Minkowski

2a. For $n \in \mathbb{N}^*$ and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$, we define the function T of $\lambda \in \mathbb{R}$ such as

$$T(\lambda) = \sum_{i=1}^n (\lambda x_i + y_i)^2$$

- (i) Show that $\forall \lambda \in \mathbb{R}, T(\lambda) \geq 0$.
- (ii) Show that $T(\lambda) = \left(\sum_{i=1}^n x_i^2\right) \lambda^2 + 2\left(\sum_{i=1}^n x_i y_i\right) \lambda + \sum_{i=1}^n y_i^2$
- (iii) Consider the equation $T(\lambda) = 0$ (assuming $\sum_{i=1}^n x_i^2 \neq 0$). What is the sign of the determinant ?
- (iv) Deduce the inequality of Cauchy-Schwartz: $\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n x_i^2\right) \left(\sum_{i=1}^n y_i^2\right)$

2b.

- (i) Using 2a, show : $\sum_{i=1}^n (x_i + y_i)^2 \leq \left(\sum_{i=1}^n x_i^2\right) + 2\sqrt{\sum_{i=1}^n x_i^2} \sqrt{\sum_{i=1}^n y_i^2} + \sum_{i=1}^n y_i^2$
- (ii) Deduct the inequality of Minkowski: $\sqrt{\sum_{i=1}^n (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^n x_i^2} + \sqrt{\sum_{i=1}^n y_i^2}$

Exercise 3: Complex index of refraction and absorption coefficient

We consider a dielectric material inside which a monochromatic plane wave with a frequency in the Infrared is traveling. The frequency turns out to correspond to a vibrational mode of the crystal lattice, leading to a transfer of energy between the electromagnetic wave and the medium. The wave is hence losing energy as it is traveling, which we can approach by considering a complex dielectric constant and index of refraction. We then define:

$$\varepsilon_r = \varepsilon_1 + i\varepsilon_2 \text{ and } n = n_1 + in_2$$

Where ε_1 and ε_2 are the real and imaginary parts of the dielectric constant respectively and n_1 and n_2 are the real and imaginary parts of the index of refraction, respectively. We remind you that $n = \sqrt{\varepsilon_r}$

3a. Show that we must have:
$$\begin{cases} n_1^2 - n_2^2 = \varepsilon_1 \\ 2n_1n_2 = \varepsilon_2 \end{cases}$$

3b. Deduce that:
$$\begin{cases} n_1 = \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2})} \\ n_2 = \frac{1}{\sqrt{2}} \sqrt{(-\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2})} \end{cases}$$

3c. We consider a plane wave of angular frequency ω and vector $k = \frac{2\pi}{\lambda}n$, λ being the wavelength, traveling along the x axis: $\vec{E}(x, t) = \vec{E}_0 e^{i(kx - \omega t)}$.

Knowing that the intensity at a point x in the material is $I(x) = |\vec{E}(x, t)|^2$, show that :

$$I = I(0)e^{-\alpha x} \text{ with } \alpha = \frac{4\pi}{\lambda}n_2$$

3d. Absorption and attenuation

(i) Gallium Arsenide (GaAs) is a material widely used in optoelectronics and particularly in photovoltaics, in part because of its high absorption in the visible. At $600 \text{ nm} = 6 \times 10^{-7} \text{ m}$, $n_2 = 0.6$.

What is the penetration depth δ_{GaAs} at 600 nm, that is the length of propagation inside the materials for which $\frac{I(\delta)}{I(0)} = \frac{1}{e}$?

(ii) Light in an silica optical fiber propagates in the core with a very low attenuation of the glass at the telecommunication wavelength $\lambda = 1.55 \mu\text{m}$. Typically, amplifiers are put every 50 km of fiber to reinforce the signal. This corresponds to a loss of 90% of intensity of the signal: $\frac{I(50 \text{ km})}{I(0)} = 0.1$

Calculate what is the imaginary index of refraction of silica at $\lambda = 1.55 \mu\text{m}$?